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On the Central Derivative of Hecke L-Series

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0. INTRODUCTION

The well-known Birch and Swinnerton–Dyer conjecture gives a deep connection between the leading coefficient of the L-series and the arithmetic properties of an abelian variety. Both are very important and subtle. This paper is part of an effort to compute the analytic side explicitly in a special case. Indeed, we are interested in the central derivative of certain algebraic Hecke L-series, related to CM abelian varieties or, more precisely, to pieces of it (CM motives). The result, together with the Gross–Zagier formula proved by Zhang [Zh], would also give a new way to compute the height of certain Heegner cycles on a Kuga–Sato variety.

Let $p \equiv 3 \bmod 4$ be a prime number such that $p > 3$. Let $k \geqslant 0$ be an integer. Let $E = \mathbb{Q}(\sqrt{-p})$ and view it as a subfield of $\mathbb{C}$ such that $\sqrt{-p} = i\sqrt{p}$. Let h_p be the ideal class number of E. A canonical Hecke character of E of weight $2k+1$ is a Hecke character μ satisfying:

(1) The conductor of μ is $\sqrt{-p}\,\mathcal{O}_E$.

(2) $\mu(\bar{\mathfrak{A}}) = \overline{\mu(\mathfrak{A})}$ for an ideal $\mathfrak{A}$ relatively prime to $\sqrt{-p}\,\mathcal{O}_E$.

(3) $\mu(\alpha \mathcal{O}_E) = \pm \alpha^{2k+1}$.

There are h_p such Hecke characters, differing from each other by an ideal class character. A canonical Hecke character of weight 1 is simply called a canonical character of E.

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Canonical Hecke characters of E of weight one are very closely related to the elliptic curves $A(p)$ studied by Gross in [Gro], while canonical Hecke characters of weight $2k+1$ are closely related to a piece of the $(2k+1)$th symmetric power of $A(p)$. The second condition implies that the root number of μ is ± 1, given by $(2/p)(-1)^k$ in the canonical case. When the root number is one, the leading coefficient of the L-series is usually the special value. In the canonical case, the special value was proved to be nonvanishing [MR] and was given by a beautiful formula discovered by Rodriquez Villegas [RV]. The explicit formula was generalized by the author using a different method [Ya1]. When the root number is -1, the special value at the center is automatically zero, and the leading coefficient is most likely the central derivative. In fact, when $k=0$, Miller and the author [MY] have proved that for all $p \equiv 3 \bmod 8$, the central derivative $L'(1, \mu) \neq 0$. In this paper, we will give an explicit formula to compute the central derivative when $p \equiv 3 \bmod 8$ and k is even. The other case where $p \equiv 7 \bmod 8$ and k is odd can be treated similarly. In fact, this method works for its quadratic twists too, but we stick to one special case to keep it simple.

From now on, we assume that $p \equiv 3 \bmod 8$ and $k \geqslant 0$ is an even integer. So the root number of μ is $(-1)^k (2/p) = -1$, and the central L-value $L(k+1, \mu) = 0$ automatically. The purpose of this paper is to give a formula for the central derivative $L'(k+1, \mu)$. In fact, our formula is about the partial L-series. For each ideal class C of E, we can define the partial L-series

$$L(s, \mu, C) = \sum_{\mathfrak{A} \in C,\ \text{integral}} \mu(\mathfrak{A})(N\mathfrak{A})^{-s}. \tag{0.1}$$

Obviously,

$$L(s, \mu) = \sum_{C} L(s, \mu, C) \qquad \text{and} \qquad L(s, \mu\xi, C) = \xi(C)\, L(s, \mu, C), \tag{0.2}$$

where ξ is an ideal class character of E. Moreover, $L(s, \mu, C)$ is characterized by (0.2).

For an ideal class C, we choose a primitive integral ideal $\mathfrak{A}$ in C relatively prime to $2p$. Write

$$\mathfrak{A} = \left[a, \frac{b+\sqrt{-p}}{2} \right] \tag{0.3}$$

with $a = N\mathfrak{A} > 0$ and $b^2 \equiv -p \bmod 4ap$. Let $\tau_{\mathfrak{A}} = (b+\sqrt{-p})/2a$.

Let $\varepsilon=(-p,\ )_{\mathbb{A}}$ be the quadratic Hecke character of $\mathbb{Q}$ associated to $E/\mathbb{Q}$ by class field theory. Let

$$\Lambda(s,\varepsilon)=L(s,\varepsilon_\infty)\,L(s,\varepsilon),\qquad \text{where}\quad L(s,\varepsilon_\infty)=\pi^{-(s+1)/2}\,\Gamma\left(\frac{s+1}{2}\right). \tag{0.4}$$

Define $\rho(n)$ for an integer $n>0$ via

$$\zeta_E(s)=\sum_{n=1}^{\infty}\rho(n)\,n^{-s}, \tag{0.5}$$

where ζ_E is the Dedekind zeta function of E. Finally, we define two functions

$$C_k(t)=\sum_{m=0}^{k}\binom{k}{m}\frac{t^m}{m!} \tag{0.6}$$

and, for $t>0$,

$$\beta_k(t)=\int_1^\infty e^{-tu}(u-1)^k\,u^{-k-1}\,du. \tag{0.7}$$

These are two "basic" solutions of the differential equation

$$tC''(t)+(1+t)\,C'(t)-kC(t)=0.$$

THEOREM 0.1. *Let the notation be as above. Then*

$$\begin{aligned}L'(k+1,\mu,C)=\frac{\pi\mu(\mathfrak{A})}{\sqrt{p}\,a^{k+1}}\Bigg[&h_p\left(\log\frac{p}{2a}+2\frac{\Lambda'(1,\varepsilon)}{\Lambda(1,\varepsilon)}+\sum_{j=1}^{k}\frac{1}{j}\right)\\&-2\sum_{n>0}a_nC_k\left(-\frac{2\pi n}{a\sqrt{p}}\right)e^{(2\pi in/p)\,\tau_{\mathfrak{A}}}\\&-2\sum_{n<0}\rho(-n)\,\beta_k\left(-\frac{2\pi n}{a\sqrt{p}}\right)e^{(2\pi in/p)\,\tau_{\mathfrak{A}}}\Bigg].\end{aligned}$$

Here

$$a_n=\sum_{q\text{ inert}}(\operatorname{ord}_q n+1)\,\rho(n/q)\log q+(\operatorname{ord}_p n+\tfrac{1}{2})\,\rho(n)\log p.$$

The sum $\sum_{j=1}^{k}(1/j)$ should be treated as zero when $k=0$.

We remark that the sum in the bracket is the special value of a non-holomorphic modular form at a CM point $\tau_{\mathfrak{A}}$. We also remark that at most one term in a_n is nonzero and that $a_n+\frac{1}{2}\rho(n)\log p$ has an interesting arithmetic interpretation [KRY, Section 5]. When C is trivial, we can do a little bit more.

THEOREM 0.2. *Let μ be a canonical Hecke character of weight one. Then the partial central derivative is*

$$\begin{aligned}(1) \quad L'(1, \mu, \text{trivial}) = \frac{\pi}{\sqrt{p}} \Bigg[& h_p \left(\log \frac{p}{2} + 2 \frac{\Lambda'(1, \varepsilon)}{\Lambda(1, \varepsilon)} \right) \\ & + 2 \sum_{n>0} (-1)^{n-1} a_n e^{-n\pi/\sqrt{p}} \\ & 2 \sum_{n<0} (-1)^{n-1} \rho(-n)\, \beta_0 \left(-\frac{2\pi n}{\sqrt{p}} \right) e^{-n\pi/\sqrt{p}} \Bigg] \end{aligned} \tag{0.8}$$

and

$$\begin{aligned}(2) \quad L'(1, \mu, \text{trivial}) = \frac{4\pi}{\sqrt{p}} \Bigg[& h_p \left(\log p + 2 \frac{\Lambda'(1, \varepsilon)}{\Lambda(1, \varepsilon)} \right) - 2 \sum_{n>0} a_n e^{-4\pi n/\sqrt{p}} \\ & - 2 \sum_{n<0} \rho(-n)\, \beta_0 \left(-\frac{8\pi n}{\sqrt{p}} \right) e^{-4\pi n/\sqrt{p}} \\ & - 2 \log 2 \sum_{n>0} \rho(n)\, e^{-2\pi n/\sqrt{p}} \Bigg]. \end{aligned} \tag{0.9}$$

This paper is organized as follows. In Section 1 we use a result in [Ya1] to write the Hecke L-series as the sum of certain "incoherent" Eisenstein series valued at some CM points. In Section 2 we prove some preliminary local results for Section 3. In Section 3 we use Kudla's idea [Ku] to compute the central derivative of the Eisenstein series in Section 1 and prove Theorem 0.1. It might be worthwhile to point out that the sections used in the Eisenstein series are not standard at bad primes. The same phenomenon also occurs in Kudla and Rapoport's work in a higher dimension case [KR]. In Section 4, we prove Theorem 0.2. In the process, we also prove a functional equation of the Eisenstein series involved with respect to τ (instead of s).

1. SETUP

Let $W = E$ with the skew-Hermitian form $\langle x, y \rangle = \delta x \bar{y}$, and let $W + W_-$ be its doubling. Let $G = G(W) = U(1) = E^1$ and $H = G(W + W_-) = U(1, 1)$ be the corresponding isometry groups acting on the right. Then one has a canonical embedding

$$i: G \times G \to H, \qquad (x_1, x_2)\, i(g_1, g_2) = (x_1 g_1, x_2 g_2). \tag{1.1}$$

Let $W^d=\{(w, w): w\in W\}$ and $W_d=\{(w, -w): w\in W\}$. Then $W+W_-$ has the standard complete polarization

$$W+W_-=W_d\oplus W^d$$

and the standard E-basis $e=(1/2\delta)(1, -1)$ and $f=(1, 1)$. With respect to the standard basis, the map (1.1) is given by

$$i(g_1, g_2)=\frac{1}{2}\begin{pmatrix} g_1+g_2 & \frac{1}{2\delta}(g_1-g_2) \\ 2\delta(g_1-g_2) & g_1+g_2 \end{pmatrix}. \tag{1.2}$$

We will write $i(g)$ for $i(g, 1)$ in this paper.

Let P be the stabilizer of W^d in H (the standard Siegel parabolic subgroup of H). Then P has the Levi decomposition $P=NM$ where, with respect to the standard complete polarization,

$$\begin{aligned} N&=\left\{n(b)=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}: b\in\mathbb{Q}\right\},\\ M&=\left\{m(a)=\begin{pmatrix}a & 0\\ 0 & \bar{a}^{-1}\end{pmatrix}: a\in E^*\right\}. \end{aligned} \tag{1.3}$$

One has

$$H=P\times i(G). \tag{1.4}$$

Let χ_{can} be a fixed canonical Hecke character of E (of weight 1), and let $\chi=\chi_{\mathrm{can}}\|_{\mathbb{A}}^{1/2}$ be its unitary counterpart. Then there is a unique character η of $G_{\mathbb{A}}$ such that $\mu=\chi\tilde{\eta}\|_{\mathbb{A}}^{-k-1/2}$, where $\tilde{\eta}(z)=\eta(z/\bar{z})$ is a Hecke character of E. So

$$L(s+k+1, \mu, C)=L(s+\tfrac{1}{2}, \chi\tilde{\eta}, C). \tag{1.5}$$

Let $I(s, \chi)=\mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}\chi\|^s=\bigotimes' I(s, \chi_v)$ be the degenerate (induced from a character of a maximal parabolic subgroup) principal series representation. Given a function $\Phi=\prod\Phi_v\in I(s, \chi)$, one defines the Eisenstein series

$$E(h, s, \Phi)=\sum_{\gamma\in P(\mathbb{Q})\backslash H(\mathbb{Q})}\Phi(\gamma h, s) \tag{1.6}$$

where $h \in H(\mathbb{A})$. It is absolutely convergent for $\operatorname{Re} s \gg 0$ and has a meromorphic continuation to the whole complex s-plane. We normalize the Eisenstein series

$$E^*(h, s) = \Lambda(2s+1, \varepsilon)\, E(h, s). \tag{1.7}$$

Let $\Phi = \prod \Phi_q \in I(s, \chi)$ be the section $\Phi_{\bar{\eta}}$ defined in [Ya1, Theorem 1.11]. More specifically, when q is nonsplit,

$$\Phi_q(n(b)\, m(a)\, i(g)) = \chi_q(a)\, |a\bar{a}|_q^{s+1}\, \bar{\eta}(g). \tag{1.8}$$

When q is split, Φ_q is characterized by (after identifying G_q with $\mathbb{Q}_q^*$ as in [Ya1, Section 1.2])

$$\begin{aligned}\Phi_q(i(g)) = {} & \chi_q(g)\, |g|^{s+1/2} \operatorname{char}(\mathbb{Z}_q)(g) \\ & + \chi_q(g)\, |g|^{-s-1/2} \operatorname{char}(q\mathbb{Z}_q)(1/g).\end{aligned} \tag{1.9}$$

Since we will stick to the same Φ throughout this paper, we will drop Φ in the notation from now on.

Proposition 1.1. *For each ideal class C, choose an ideal $a_C \in E_{\mathbb{A}}^*$ such that the corresponding ideal is $\mathfrak{A} \in C$, primitive and relatively prime to $2p$. Let g_C be the image of a_C in $G_{\mathbb{A}}$ under the map $z \mapsto z/\bar{z}$. Then*

$$L'(k+1, \mu) = \frac{\pi}{2} \sum_{C \in CL(E)} \tilde{\eta}(\mathfrak{A})\, E^{*,\prime}(i(g_C), 0).$$

Proof. Locally, we choose the Haar measure on G_q such that $\operatorname{meas}(G_q) = 1$ when q is nonsplit and $\operatorname{meas}(\mathbb{Z}_q^*) = 1$ when q is split. We take the product measure on $G(\mathbb{A})$ and the quotient measure on $[G] = G(\mathbb{Q}) \backslash G(\mathbb{A})$. Then [Ya1, Theorem 1.1] implies

$$\frac{L(s+k+1, \mu)}{L(2s+1, \varepsilon)} = \frac{1}{\operatorname{meas}[G]} \int_{[G \times G]} E(i(g_1, g_2), s)\, \eta(g_1)(\chi\eta)^{-1}(g_2)\, dg_1\, dg_2 .$$

Since $E(i(g_1, g_2), s) = E(i(g_1 g_2^{-1}, 1), s)\, \chi(g_2)$, a substitution gives

$$\frac{L(s+k+1, \mu)}{L(2s+1, \varepsilon)} = \int_{[G]} E(i(g), s)\, \eta(g)\, dg.$$

Next, note that $\coprod g_C^{-1} U^1$ is a fundamental domain for $[G]$ (see [RVY, Proof of Proposition 1.3]) where

$$U^1 = \prod_{q \text{ inert}} E_q^1 \times \prod_{q \text{ split}} \mathbb{Z}_q^* \times \{ g \in E_p^1 : g \equiv 1 \bmod \delta \}.$$

So

$$\frac{L(s+k+1,\mu)}{L(2s+1,\varepsilon)}=\sum_{C}\int_{g_C^{-1}U^1} E(i(g),s)\,\eta(g)\,dg$$

$$=\sum_{C}\eta(g_C)\int_{U^1} E(i(g_C g),s)\,\eta(g)\,dg.$$

Finally, one has by definition

$$\Phi(hi(g),s)=\bar{\eta}(g)\,\Phi(h,s)$$

for all $h\in H(\mathbb{A})$ and $g\in U^1$. Therefore

$$\frac{L(s+k+1,\mu)}{L(2s+1,\varepsilon)}=\sum_{C}\tilde{\eta}(\mathfrak{A})\,E(i(g_C),s)\int_{U^1} dg=\frac{1}{2}\sum_{C}\tilde{\eta}(\mathfrak{A})\,E(i(g_C),s).$$

Here $\frac{1}{2}$ appears because the subgroup $\{g\in E_p^1 : g\equiv 1 \bmod \delta\}$ is of index 2 in $G_p=E_p^1$. Multiplying both sides by $\Lambda(2s+1,\varepsilon)$, one has

$$L(2s+1,\varepsilon_\infty)\,L(s+k+1,\mu)=\tfrac{1}{2}\sum_{C}\tilde{\eta}(\mathfrak{A})\,E^*(i(g_C),s). \tag{1.10}$$

Recall that $L(1,\varepsilon_\infty)=\pi^{-1}$. Taking the derivative at $s=0$, one gets the desired formula.

To motivate the computation in the next section, we recall that

$$E^*(h,s)=\sum_{t\in\mathbb{Q}} E_t^*(h,s), \tag{1.11}$$

where

$$E_0^*(h,s)=\Phi^*(h,s)+\prod_q W_{0,q}^*(h,s) \tag{1.12}$$

is the constant term and

$$E_t^*(h,s)=\prod_q W_{t,q}^*(h,s) \tag{1.13}$$

is the tth (normalized) Fourier coefficient of E for $t\neq 0$. Here $\Phi^*(h,s)=\Lambda(2s+1,\varepsilon)\,\Phi(h,s)$ and

$$W_{t,q}^*(h,s)=L(2s+1,\varepsilon_q)\,W_{t,q}(h,s) \tag{1.14}$$

is the normalized local Whittaker functions (including $t=0$).

2. LOCAL WHITTAKER FUNCTIONS

The purpose of this section is to compute the local Whittaker functions and their central derivative. We will write $W_{t,q}(s)$ for $W_{t,q}(1, s)$ from now on and drop the subscript q when no confusion will occur. For example, χ will mean χ_q in this section. We start with the nonsplit case.

LEMMA 2.1. *Assume that q is nonsplit in E. Then one has for $t \in \mathbb{Q}_q$,*

$$W_{t,q}(s) = \int_{\mathbb{Q}_q} (\chi\tilde{\eta})^{-1}\left(b + \frac{1}{2\delta}\right)\left|b + \frac{1}{2\delta}\right|_E^{-s-1/2} \psi(-tb)\, db.$$

Proof. Since q is nonsplit, $H_q = P_q \times i(G_q)$. So one can write

$$wn(b) = n(x)\, m(y)\, i(g).$$

Then

$$g = \bar{y}/y \qquad \text{and} \qquad b + \frac{1}{2\delta} = \bar{y}^{-1} g.$$

So

$$y = \frac{1}{b + 1/2\delta}, \qquad g = \frac{b + 1/2\delta}{b - 1/2\delta}.$$

Now it is clear that

$$\begin{aligned} W_{t,q}(s) &= \int_{\mathbb{Q}_q} \Phi(wn(b))\, \psi(-tb)\, db \\ &= \int_{\mathbb{Q}_q} \chi(y)\, \bar{\eta}(\bar{y}/y)\left|b + \frac{1}{2\delta}\right|_E^{-s-1/2} \psi(-tb)\, db \\ &= \int_{\mathbb{Q}_q} (\chi\tilde{\eta})^{-1}\left(b + \frac{1}{2\delta}\right)\left|b + \frac{1}{2\delta}\right|_E^{-s-1/2} \psi(-tb)\, db, \end{aligned}$$

as stated.

PROPOSITION 2.2. *When $q \nmid 2p\infty$ is inert, $W^*_{t,q}(s) = 0$ unless $t \in \mathbb{Z}_q$. Assume that $t \in \mathbb{Z}_q$, and set $X = q^{-2s}$. Then*

$$W^*_{t,q}(s) = \sum_{k=0}^{\operatorname{ord}_q t} (-X)^k. \tag{1}$$

In particular,

$$W^*_{0,q}(s) = L(2s, \varepsilon_q).$$

(2) *When* $0 \neq t \in \mathbb{Z}_q$, $W^*_{t,q}(0)$ *equals* 1 *or* 0 *depending on whether* $\mathrm{ord}_q\, t$ *is even or odd. When* $\mathrm{ord}_q\, t$ *is odd,*

$$W^{*,\prime}_{t,q}(0) = (\mathrm{ord}_q\, t + 1) \log q.$$

PROPOSITION 2.3. (1)

$$W^*_{0,\infty}(s) = -i2^{2s}p^s L(2s, \varepsilon_\infty) \prod_{j=1}^{k} \frac{j-s}{j+s}.$$

(2) *When* $t > 0$, *one has*

$$W^*_{t,\infty}(0) = -2ie^{-\pi t/\sqrt{p}}\, C_k\left(-\frac{2\pi t}{\sqrt{p}}\right)$$

where $C_k(t)$ *is defined by* (0.6).

(3) *When* $t < 0$, $W^*_{t,\infty}(0) = 0$ *and*

$$W^{*,\prime}_{t,\infty}(0) = -2ie^{-\pi t/\sqrt{p}}\, \beta_k(2\pi\, |t|/\sqrt{p}),$$

where $\beta_k(a)$ *is as defined by* (0.7).

The above two propositions can be proved by standard calculations using Lemma 2.1 (see [KRY, Proposition 2.6], for example) and are left to the reader. Recall [Ya1] that Φ_q is the standard section in $I(s, \chi_q)$ associated to $\mathrm{char}(\mathcal{O}_{E_q}) \in S(E_q)$ with respect to a certain Weil representation of $U(1,1) \times U(1)$ when $q \nmid 2p$. When $q \mid 2p$, $i(g) \notin H(\mathbb{Z}_p)$ for $g \in G(\mathbb{Q}_q)$, and this prevents Φ_q from being a standard section.

PROPOSITION 2.4. $W^*_{t,2}(s) = 0$ *unless* $t \in \mathbb{Z}_2$. *Assume* $t \in \mathbb{Z}_2$, *and set* $X = 2^{-2s}$.

$$(1) \qquad W^*_{t,2}(s) = \begin{cases} -1 & \text{if } t \in \mathbb{Z}_2^*, \\ \sum_{k=0}^{\mathrm{ord}_2 t} (-X)^k & \text{if } t \in 2\mathbb{Z}_2. \end{cases}$$

In particular,

$$W^*_{0,2}(s) = L(2s, \varepsilon_2).$$

(2) *When* $0 \neq t \in 2\mathbb{Z}_2$, $W^*_{t,2}(0)$ *is* 1 *or* 0 *depending on whether* $\mathrm{ord}_2\, t$ *is even or odd. Moreover, when* $\mathrm{ord}_2\, t$ *is odd, one has*

$$W^{*,\prime}_{t,2}(0) = (\mathrm{ord}_2\, t + 1) \log 2.$$

Proof. First note that when $b \in \frac{1}{2}\mathbb{Z}_2^*$, one has

$$b + \frac{1}{2\delta} = \frac{1 + 2b\delta}{2\delta} \in \mathbb{Q}_2^*.$$

So Lemma 2.1 implies

$$\begin{aligned} W_{t,2}(s) &= \int_{\mathbb{Z}_2} \chi\tilde{\eta}(2\delta)\, |2|_2^{2s+1}\, \psi(-tb)\, db + \int_{(1/2)\,\mathbb{Z}_2^*} \psi(-tb)\, db \\ &\quad + \sum_{k=2}^{\infty} (-X)^k \int_{\mathbb{Z}_2^*} \psi(-2^{-k} tb)\, db \\ &= -\tfrac{1}{2} X\, \mathrm{char}(\mathbb{Z}_2)(t) + 2\, \mathrm{char}(2\mathbb{Z}_2)(t) - \mathrm{char}(\mathbb{Z}_2)(t) \\ &\quad + \sum_{k=2}^{\infty} (-X)^k \left(\mathrm{char}(2^k\mathbb{Z}_2) - \tfrac{1}{2}\, \mathrm{char}(2^{k-1}\mathbb{Z}_2)\right)(t). \end{aligned}$$

So $W_{t,2}(s) = 0$ unless $t \in \mathbb{Z}_2$. When $t \in \mathbb{Z}_2^*$,

$$W_{t,2}(s) = -\tfrac{1}{2} X - 1 = -L(2s+1, \varepsilon_2)^{-1},$$

as expected. When $r = \mathrm{ord}_2\, t > 0$,

$$\begin{aligned} W_{t,2}(s) &= \tfrac{1}{2}(-X) + 1 + \tfrac{1}{2} \sum_{k=2}^{r} (-X)^k - \tfrac{1}{2}(-X)^{r+1} \\ &= (1 + \tfrac{1}{2} X) \sum_{k=0}^{r} (-X)^k. \end{aligned}$$

So

$$W^*_{t,2}(s) = \sum_{k=0}^{r} (-X)^k.$$

The rest follows easily.

PROPOSITION 2.5. *$W^*_{t,p}(s)=0$ unless $t\in\mathbb{Z}_p$. Assume $t\in\mathbb{Z}_p$ and set $X=p^{-s}$.*

(1)
$$W^*_{t,p}(s)=\frac{-i}{\sqrt{p}}\,(X-\varepsilon_p(t)\,X^{2(\mathrm{ord}_p t+1)})$$

for $t\neq 0$, and

$$W^*_{0,p}(s)=\frac{-i}{\sqrt{p}}\,X.$$

(2) *When $0\neq t\in\mathbb{Z}_p$,*

$$W^*_{t,p}(0)=\begin{cases}\dfrac{-2i}{\sqrt{p}} & \text{if } \varepsilon_p(t)=-1,\\ 0 & \text{if } \varepsilon_p(t)=1.\end{cases}$$

Moreover, when $\varepsilon_p(t)=1$,

$$W^{*,\prime}_{t,p}(0)=\frac{-2i}{\sqrt{p}}\left(\mathrm{ord}_p\, t+\frac{1}{2}\right)\log p.$$

Proof. Recall that $\delta=\sqrt{-p}$ is a uniformizer of E_p. Simple calculation using [Ya3, Lemma 2.4] gives

$$\begin{aligned}W_{t,p}(s)&=\int_{\mathbb{Z}_p}(\chi\tilde{\eta})(2\delta)\,|2\delta|_E^{s+1/2}\,\psi(-tb)\,db+\sum_{k=1}^{\infty}X^{2k}\int_{\mathbb{Z}_p^*}\varepsilon(b)\,\psi(-p^{-k}tb)\,db\\&=-ip^{-1/2}X\,\mathrm{char}(\mathbb{Z}_p)(t)\\&\quad+\sum_{k=1}^{\infty}X^{2k}p^{-1/2}\varepsilon_p(-p^k t)\,\varepsilon(\tfrac{1}{2},\varepsilon_p,\psi_p)\,\mathrm{char}(p^{k-1}\mathbb{Z}_p^*)(t).\end{aligned}$$

So $W_{t,p}(s)=0$ unless $t\in\mathbb{Z}_p$. When $r=\mathrm{ord}_p\,t\geqslant 0$, recall that $\varepsilon_p(-p^k)=-1$ and

$$\varepsilon(\tfrac{1}{2},\varepsilon_p,\psi_p)=\varepsilon(\tfrac{1}{2},\varepsilon_\infty,\psi_\infty)^{-1}=-i.$$

So one has

$$W^*_{t,p}(s)=W_{t,p}(s)=\frac{-i}{\sqrt{p}}\,(X-\varepsilon_p(t)\,X^{2(r+1)})$$

as stated. The other claims now follow easily.

Now we assume that q is split in E. We make identifications as in [Ya1, Section 1.2]. In particular, G_q is identified with $\mathbb{Q}_q^*$ via $(z,z^{-1})\mapsto z$, and

H_q is identified with $GL_2(\mathbb{Q}_q)$. Also, χ_q is identified with $\chi_w = \chi_{\bar{w}}^{-1}$ and is viewed as a character of $\mathbb{Q}_q^*$. With respect to the standard basis specified in [Ya1, Section 1.2], one has

$$i(g) = \frac{1}{2}\begin{pmatrix} g+1 & \frac{1}{2x_q}(g-1) \\ 2x_q(g-1) & g+1 \end{pmatrix}. \tag{2.1}$$

Here we have written $\delta = (x_q, -x_q) \in E_q = \mathbb{Q}_q^2$ with $x_q \in \mathbb{Z}_q^*$.

Proposition 2.6. *Let $g \in \mathbb{Q}_q^*$, and let $n = |\mathrm{ord}_q\, g|$. Let $r = \mathrm{ord}_q\, t$ and set $X = q^{-s}$. Then $W_{t,q}^*(i(g), s) = 0$ unless $r \geqslant -n$. In such a case,*

$$W_{t,q}^*(i(g), s) = \chi_q(g)\, q^{-n/2} \psi_q\left(\mp \frac{t}{2x_q}\right) \sum_{k=0}^{r+n} X^{2k-n},$$

where $\mp$ depends on whether $g \in \mathbb{Z}_q$ or not. In particular,

$$W_{0,q}^*(i(g), s) = \chi_q(g)\, q^{-n/2}\, X^{-n} L(2s, \varepsilon_q).$$

Proof. Write (for $b \neq \pm 1/2x_q$)

$$wn(b) = n(x)\, m(y_1, y_2)\, i(g_b), \tag{2.2}$$

with $x \in \mathbb{Q}_q$, $y_i \in \mathbb{Q}_q^*$, and $g_b \in G(\mathbb{Q}_q) = \mathbb{Q}_q^*$. Here

$$m(y_1, y_2) = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}.$$

Then ($i(g_b)$ fixes $\pm 1/2x_q$)

$$\begin{aligned} y_1 &= \left(b + \frac{1}{2x_q}\right)^{-1}, \\ y_2 &= b - \frac{1}{2x_q}, \\ g_b &= (y_1\, y_2)^{-1} = \frac{b + 1/2x_q}{b - 1/2x_q}. \end{aligned} \tag{2.3}$$

It is easy to check that $b \mapsto g_b$ is a bijection of $\mathbb{P}^1(\mathbb{Q}_q)$ onto itself. Moreover, for $n \geqslant 0$,

$$g_b \in q^{-n}\mathbb{Z}_q \qquad \text{if and only if} \quad b \notin \frac{1}{2x_q} + q^{n+1}\mathbb{Z}_q, \tag{2.4}$$

and for $n>0$,

$$g_b \in q^n\mathbb{Z}_q \qquad \text{if and only if} \quad b \in -\frac{1}{2x_q} + q^n\mathbb{Z}_q. \tag{2.5}$$

So

$$\begin{aligned}\Phi_q(wn(b)\, i(g)) &= \chi_q(y_1\, y_2)\, |y_1/y_2|_q^{s+1/2}\, \Phi_q(i(g_b\, g), s)\\ &= \chi_q^{-1}(g_b)\, |y_1/y_2|_q^{s+1/2}\, \Phi_q(i(g_b\, g), s).\end{aligned}$$

When $g \in \mathbb{Z}_q$, $n = \operatorname{ord}_q g$. So (2.4) implies that $g_b\, g \in \mathbb{Z}_q$ if and only if $b \notin 1/2x_q + q^{n+1}\mathbb{Z}_q$. One has then by (1.9)

$$\begin{aligned}&\Phi_q(wn(b)\, i(g), s)\\ &= \chi_q(g)\, |g|^{-s-1/2} \begin{cases} 1 & \text{if} \quad b \in \dfrac{1}{2x_q} + q^{n+1}\mathbb{Z}_q,\\ \left|b - \dfrac{1}{2x_q}\right|^{-2s-1} |g|^{2s+1} & \text{if} \quad b \notin \dfrac{1}{2x_q} + q^{n+1}\mathbb{Z}_q.\end{cases}\end{aligned}$$

Therefore,

$$\begin{aligned}&W_{t,q}(i(g), s)\, \chi^{-1}(g)\, |g|^{s+1/2}\, \psi\left(\frac{t}{2x_q}\right)\\ &= \psi\left(\frac{t}{2x_q}\right)\left(\int_{1/2x_q + q^{n+1}\mathbb{Z}_q} \psi(-tb)\, db\right.\\ &\qquad \left. + |g|^{2s+1} \int_{b \notin 1/2x_q + q^{n+1}\mathbb{Z}_q} \left|b - \frac{1}{2x_q}\right|^{-2s-1} \psi(-tb)\, db\right)\\ &= q^{-n-1}\, \operatorname{char}(q^{-n-1}\mathbb{Z}_q)(t) + X^{2n} q^{-n} \sum_{k=-n}^{\infty} X^{2k} \int_{\mathbb{Z}_q^*} \psi(-tb)\, db\\ &= q^{-n}\left[\frac{1}{q}\operatorname{char}(q^{-n-1}\mathbb{Z}_q)\right.\\ &\qquad \left. + X^{2n} \sum_{k=-n}^{\infty} X^{2k} \left(\operatorname{char}(q^k\mathbb{Z}_q) - \frac{1}{q}\operatorname{char}(q^{k-1}\mathbb{Z}_q)\right)\right](t).\end{aligned}$$

So $W_{t,q}(i(g), s) = 0$ unless $r = \operatorname{ord}_q t \geqslant -n-1$. When $r = -n-1$, one has

$$W_{t,q}(i(g), s)\, \chi^{-1}(g)\, |g|^{s+1/2}\, \psi\left(\frac{t}{2x_q}\right) = q^{-n}(q^{-1} - X^{2n} q^{-1} X^{-2n}) = 0.$$

When $r \geqslant -n$, one has

$$\begin{aligned} W_{t,q}(i(g), s)\,\chi^{-1}(g)\,|g|^{s+1/2}\,\psi\left(\frac{t}{2x_q}\right) &= q^{-n}\left[q^{-1} + (1-q^{-1})\sum_{k=0}^{r+n} X^{2k} - q^{-1}X^{2r+2n+2}\right] \\ &= q^{-n}L(2s+1, \varepsilon_q)\sum_{k=0}^{r+n} X^{2k}. \end{aligned}$$

So

$$W^*_{t,q}(i(g), s) = \chi(g)\,q^{-n/2}\,\psi\left(-\frac{t}{2x_q}\right) X^{-n}\sum_{k=0}^{r+n} X^{2k}$$

as desired. The case $g \notin \mathbb{Z}_q$ can be proved similarly using (2.5) and is left to the reader.

3. PROOF OF THEOREM 0.1

We start by choosing g_C more explicitly. For an ideal class C of E, choose a primitive ideal $\mathfrak{A} \in C$ relatively prime to $2p$. Write

$$\mathfrak{A} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$$

where $\mathfrak{P}_i$ are prime ideals of E. Then $p_i = N\mathfrak{P}_i$ are prime numbers split in E. Recall the lattice decomposition (0.3) of $\mathfrak{A}$. Then $a = \prod p_i^{e_i}$. Let w_i be the place of E corresponding to $\mathfrak{P}_i$ and let $\bar{w}_i$ be the place corresponding to $\bar{\mathfrak{P}}_i$. Write $\delta = (x_i, -x_i) \in E_{p_i} = E_{w_i} \oplus E_{\bar{w}_i}$. Then

$$b \equiv -x_i \bmod p_i^{e_i} \tag{3.1}$$

since

$$\mathrm{ord}_{w_i}(b + x_i) = \mathrm{ord}_{w_i}(b + \delta) \geqslant \mathrm{ord}_{\mathfrak{P}_i} \mathfrak{A} = e_i.$$

Define $a_C = (a_w) \in E^*_{\mathbb{A}}$ such that $a_w = p_i^{e_i}$ when $w = w_i$ and $a_w = 1$ otherwise. Then the associated ideal of a_C is $\mathfrak{A}$. Let $g_C = (g_q)$ be the image of a_C in $E^1_{\mathbb{A}}$. So $g_q = 1$ unless $q = p_i$, in which case $g_{p_i} = p^{e_i}$ under our specific identification.

As in [KRY], we define $\rho_q(n)$, for a prime number q and an integer n, to be $\rho(q^{\mathrm{ord}_q n})$. We also define $\rho_\infty(n)$ to be 1 or 0 depending on whether n is positive or not. Finally, we define $\rho(n) = 0$ when $n < 0$. Then $\rho_p(n) = 1$ and

$$\rho(n) = \prod_q \rho_q(n). \tag{3.2}$$

PROPOSITION 3.1. (1) *Assume that $q \nmid a$ is split. Then $W^*_{t,q}(i(g_C), s) = 0$ unless $t \in \mathbb{Z}_q$. Moreover,*

$$W^*_{t,q}(i(g_C), 0) = (\operatorname{ord}_q t + 1)\operatorname{char}(\mathbb{Z}_q)(t).$$

(2) *When $q = p_i \mid a$, $W^*_{t,q}(i(g_C), s) = 0$ unless $t \in p_i^{-e_i}\mathbb{Z}_{p_i}$. In such a case,*

$$W^*_{t,q}(i(g_C), 0) = \frac{\psi_{p_i}(-t/2x_{p_i})\,\chi(\mathfrak{P}_i^{e_i})}{N\mathfrak{P}_i^{e_i/2}}(\operatorname{ord}_{p_i} t + e_i + 1)\operatorname{char}(p_i^{-e_i}\mathbb{Z}_{p_i})(t).$$

(3) *For $0 \neq t \in \mathbb{Z}$, one has*

$$\tilde{\eta}(\mathfrak{A}) \prod_{q \in S_{sp}} W^*_{ta,q}(i(g_C), 0) = \frac{\mu(\mathfrak{A})}{a^{k+1}}(-1)^t e^{\pi i t b/ap} \prod_{q \in S_{sp}} \rho_q(t).$$

(4) *For $t = 0$, one has*

$$\tilde{\eta}(\mathfrak{A}) \prod_{q \in S_{sp}} W^*_{0,q}(i(g_C), s) = \frac{\mu(\mathfrak{A})}{a^{k+1}} a^s \prod_{q \in S_{sp}} L(2s, \varepsilon_q).$$

Proof. Claims (1), (2), and (4) follow from Proposition 2.6 immediately. For (3), first note that for an integer $t \neq 0$,

$$\rho_q(t) = \operatorname{ord}_q t + 1 = \begin{cases} \operatorname{ord}_q t/a + 1 & \text{if } q \nmid a, \\ \operatorname{ord}_q t/a + e_i + 1 & \text{if } q = p_i \mid a. \end{cases} \tag{3.3}$$

So (1) and (2) imply (recall that $\mu = \chi\tilde{\eta}\|\,\|_{\mathbb{A}}^{-k-1/2}$)

$$\tilde{\eta}(\mathfrak{A}) \prod_{q \in S_{sp}} W^*_{t/a,q}(i(g_C), 0) = \frac{\mu(\mathfrak{A})}{a^{k+1}} \prod_{q \in S_{sp}} \rho_q(t) \prod_i \psi_{p_i}\left(-\frac{t}{2x_i a}\right).$$

Choose integers $\tilde{2}$, $\tilde{x}_i$, and $\tilde{a}_i$ such that

$$2\tilde{2} \equiv 1 \bmod pa,$$

and

$$x_i \tilde{x}_i \equiv 1 \bmod p_i^{e_i}, \qquad a_i \tilde{a}_i \equiv 1 \bmod p_i^{e_i}.$$

Here $a_i = ap^{-e_i}$. Then

$$\psi_{p_i}\left(-\frac{t}{2x_i a}\right) = e\left(\frac{t\tilde{2}\tilde{x}_i\tilde{a}_i}{p^{e_i}}\right)$$

and

$$\prod_i \psi_{p_i}\left(-\frac{t}{2x_i a}\right) = e\left(\frac{t\widetilde{2}\widetilde{b}}{a}\right).$$

Here $e(z) = e^{2\pi i z}$ and $\widetilde{b} = \sum_i \tilde{x}_i a_i \tilde{a}_i \equiv \tilde{x}_i \bmod p_i^{e_i}$. Since $b \equiv -x_i \bmod p^{e_i}$ for all i, and $b^2 \equiv -p \bmod 4ap$, one has

$$b\widetilde{b} \equiv -1 \bmod a,$$

and

$$p\widetilde{b} \equiv -b^2\widetilde{b} \equiv b \bmod ap.$$

So

$$\prod_i \psi_{p_i}\left(-\frac{t}{2x_i a}\right) = e\left(\frac{tb\widetilde{2}}{ap}\right) = e^{\pi i t b(2\widetilde{2}/ap)} = (-1)^{tb}\, e^{\pi i t b/ap} = (-1)^t\, e^{\pi i t b/ap}.$$

This proves (3).

The following lemma follows from Propositions 2.2–2.5 and 3.1 immediately.

LEMMA 3.2. *$E_t^*(i(g_C), s) = 0$ unless $t \in \frac{1}{a}\mathbb{Z}$.*

Define

$$\Lambda(s, \mu, C) = \tfrac{1}{2}\,\tilde{\eta}(\mathfrak{A})\, E^*(i(g_C), s) \tag{3.4}$$

and

$$\Lambda_t(s, \mu, C) = \tfrac{1}{2}\,\tilde{\eta}(\mathfrak{A})\, E^*_{t/a}(i(g_C), s). \tag{3.5}$$

Then $\Lambda_t(s, \mu, C) = 0$ unless t is an integer by Lemma 3.2, and

$$L'(k+1, \mu) = \pi \sum_C \Lambda'(0, \mu, C) \tag{3.6}$$

by Proposition 1.1.

LEMMA 3.3. *One has*

$$\Lambda'(0, \mu, C) = \sum_{q<\infty \text{ inert}} \sum_{t \in qNE^*} \Lambda_t'(0, \mu, C) + \sum_{t \in NE^*} \Lambda_t'(0, \mu, C) + \sum_{t \in -NE^*} \Lambda_t'(0, \mu, C) + \Lambda_0'(0, \mu, C).$$

Here the sums are over integers t.

Proof. This is a special case of a general principle discovered by Kudla [Ku]. The key point is the following fact: In a nondegenerate Fourier coefficient of the incoherent Eisenstein series, at least one local Whittaker function vanishes at the center, and whether it vanishes is controlled by local root numbers, the local epsilon dichotomy principle [HKS]. We give a direct proof of this lemma here to show the general principle.

For each rational integer $t \neq 0$, let $D(t)$ be the set of primes q not satisfying the local epsilon dichotomy condition

$$\varepsilon(\tfrac{1}{2}, (\chi\tilde{\eta})_q, \tfrac{1}{2}\psi_{E_q})(\chi\tilde{\eta})_q(\delta) = \varepsilon_q(t).$$

Here $\psi = \prod \psi_q$ is the "canonical" nontrivial additive character of $\mathbb{Q}_{\mathbb{A}}/\mathbb{Q}$ as in [Ya2] and $\frac{1}{2}\psi_{E_q}(z) = \psi(\frac{1}{2}\operatorname{tr}_{E_q/\mathbb{Q}_q} z)$, and $\varepsilon(\frac{1}{2}, (\chi\tilde{\eta})_q, \frac{1}{2}\psi_{E_q})$ is Tate's local root number. More concretely, (1) $p \in D(t)$ if and only if $\varepsilon_p(t) = 1$; and (2) for a prime $q \neq p$, $q \in D(t)$ if and only if $\varepsilon_q(t) = -1$. It is easy to see that $D(t)$ is a finite set of nonsplit primes of odd order. Note that $D(t) = D(t/a)$ since a is a norm from E^*. The key is that $W^*_{t/a,\, q}(0) = 0$ for every $q \in D(t)$ (which is clear from calculations in Section 2). So $E^{*,\prime}_{t/a}(i(g_C), 0) = 0$ unless $|D(t)| = 1$. For each prime q, collect the terms $\Lambda_t'(0)$ together for all t such that $D(t) = \{q\}$. Since the set

$$\{t \neq 0 : D(t) = \{q\}\} = \begin{cases} NE^* & \text{if} \quad q = p, \\ -NE^* & \text{if} \quad q = \infty, \\ qNE^* & \text{if} \quad \left(\dfrac{q}{p}\right) = -1, \end{cases}$$

one proves the lemma.

Proposition 3.4. *Let t be a nonzero integer and let $e_p(z) = e^{2\pi i z/p}$. Then*

$$\Lambda_t'(0, \mu, C) = -\frac{2}{\sqrt{p}}\frac{\mu(\mathfrak{A})}{a^{k+1}}\, e_p(t\tau_{\mathfrak{A}})$$
$$\times \begin{cases} \rho\left(\dfrac{t}{q}\right)(\operatorname{ord}_q t + 1)\, C_k\left(-\dfrac{2\pi t}{a\sqrt{p}}\right)\log q & \textit{if} \quad \rho\left(\dfrac{t}{q}\right) \neq 0, \\ \rho(t)\left(\operatorname{ord}_p t + \dfrac{1}{2}\right) C_k\left(-\dfrac{2\pi t}{a\sqrt{p}}\right)\log p & \textit{if} \quad \rho(t) \neq 0, \\ \rho(-t)\, \beta_k\left(-\dfrac{2\pi t}{a\sqrt{p}}\right) & \textit{if} \quad \rho(-t) \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Here we require q to be inert when $\rho(t/q) \neq 0$.

Proof. By Propositions 2.2 and 2.4, one has

$$W^*_{t,q}(0)=\begin{cases}\rho_q(t) & \text{if} \quad q \nmid 2\infty \text{ is inert},\\ (-1)^t\,\rho_2(t) & \text{if} \quad q=2.\end{cases} \tag{3.7}$$

Assume first that $\rho(t/q)\neq 0$ for an inert prime q. Then q is unique and $W^*_{t,q}(0)=0$. In this case, $t>0$ and $\varepsilon_p(t/a)=\varepsilon_p(q)=-1$. So

$$W^*_{t/a,\infty}(0)=-2i\rho_\infty(t)\,C_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)e^{-\pi t/a\sqrt{p}} \tag{3.8}$$

and

$$W^*_{t/a,p}(0)=\frac{-2i}{\sqrt{p}}\,\rho_p(t) \tag{3.9}$$

by Propositions 2.3 and 2.5. Also, $\prod_{l\neq q}\rho_l(t)=\rho(t/q)$ in this case. So one has by Propositions 2.2, 2.4, and 3.1 and formulae (3.7)–(3.9),

$$\begin{aligned}\Lambda_t'(0,\mu,C)&=\frac{1}{2}\tilde{\eta}(\mathfrak{A})\,E^{*,\prime}_{t/a}(i(g_C),0)\\ &=\frac{1}{2}W^{*,\prime}_{t/a,q}(0)\prod_{l\neq q,\,\text{nonsplit}}W^*_{t/a,l}(0)\cdot\tilde{\eta}(\mathfrak{A})\prod_{l\text{ split}}W^*_{t/a,l}(i(g_C),0)\\ &=\frac{1}{2}(\operatorname{ord}_q t+1)\log q(-1)^t\frac{-2i}{\sqrt{p}}(-2i)\,C_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)e^{-\pi t\sqrt{p}/ap}\\ &\quad\times\prod_{l\neq q}\rho_l(t)\frac{\mu(\mathfrak{A})}{a^{k+1}}(-1)^t\,e^{\pi itb/ap}\\ &=-\frac{2}{\sqrt{p}}\rho(t/q)(\operatorname{ord}_q+1)\log q\cdot\frac{\mu(\mathfrak{A})}{a^{k+1}}C_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)e_p(t\tau_{\mathfrak{A}})\end{aligned}$$

as stated.

Next, assume that $\rho(t)\neq 0$, i.e., t is a norm from E^*. In this case, $W^*_{t/a,p}(0)=0$. By Propositions 2.5 and 3.1 and formulae (3.7)–(3.8), one then has

$$\begin{aligned}\Lambda_t'(0,\mu,C)&=\frac{1}{2}\tilde{\eta}(\mathfrak{A})\,E^{*,\prime}_{t/a}(i(g_C),0)\\ &=\frac{1}{2}W^{*,\prime}_{t/a,p}(0)\prod_{l\neq p,\,\text{inert}}W^*_{t/a,l}(0)\cdot\tilde{\eta}(\mathfrak{A})\prod_{l\text{ split}}W^*_{t/a,l}(i(g_C),0)\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{2}\frac{-2i}{\sqrt{p}}\left(\operatorname{ord}_p t+\frac{1}{2}\right)\log p(-1)^t\left(-2iC_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)\right)\\
&\quad\times e^{-\pi t/a\sqrt{p}}\prod_{l\neq p}\rho_l(t)\frac{\mu(\mathfrak{A})}{a^{k+1}}(-1)^t e^{\pi itb/ap}\\
&=-\frac{2}{\sqrt{p}}\rho(t)\left(\operatorname{ord}_p t+\frac{1}{2}\right)\log p\cdot\frac{\mu(\mathfrak{A})}{a^{k+1}}C_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)e_p(t\tau_{\mathfrak{A}})
\end{aligned}
$$

as stated.

Finally, assume that $\rho(-t)\neq 0$, i.e., $-t$ is a norm. In this case $W^*_{t/a,\infty}(0)=0$. Also, $\varepsilon_p(t/a)=\varepsilon_p(-1)=-1$, so $W^*_{t,p}$ is given by (3.9). By Propositions 2.3 and 2.5 and formulae (3.7)–(3.9), one has

$$
\begin{aligned}
\Lambda'_t(0,\mu,C)&=\frac{1}{2}\tilde{\eta}(\mathfrak{A})\,E^{*,\prime}_{t/a}(i(g_C),0)\\
&=\frac{1}{2}W^{*,\prime}_{t/a,\infty}(0)\prod_{l<\infty,\text{ nonsplit}}W^*_{t/a,l}(0)\cdot\tilde{\eta}(\mathfrak{A})\prod_{l\text{ split}}W^*_{t/a,l}(i(g_C),0)\\
&=\frac{1}{2}(-2i)\,\beta_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)e^{-\pi t/a\sqrt{p}}(-1)^t\frac{-2i}{\sqrt{p}}\\
&\quad\times\prod_{l<\infty}\rho_l(t)\frac{\mu(\mathfrak{A})}{a^{k+1}}(-1)^t e^{\pi itb/ap}\\
&=-\frac{2}{\sqrt{p}}\rho(-t)\,\beta_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)\frac{\mu(\mathfrak{A})}{a^{k+1}}e_p(t\tau_{\mathfrak{A}}),
\end{aligned}
$$

as stated.

PROPOSITION 3.5. *For the constant term, one has*

$$
\Lambda_0(s,\mu,C)=\frac{2^s\mu(\mathfrak{A})}{2a^{k+1}p^s\prod_{j=1}^k(j+s)}\left[G\left(\frac{p}{2a},s\right)-G\left(\left(\frac{p}{2a},-s\right)\right]\right.
$$

where

$$
G(y,s)=y^s\Lambda(1+2s,\varepsilon)\prod_{J=1}^{k}(j+s).
$$

Moreover,

$$\Lambda_0'(0, \mu, C) = \frac{h_p \mu(\mathfrak{A})}{\sqrt{p}\, a^{k+1}} \left[2\, \frac{\Lambda'(1, \varepsilon)}{\Lambda(1, \varepsilon)} + \log \frac{p}{2a} + \sum_{j=1}^{k} \frac{1}{j} \right].$$

Proof. First one has by definition $\Phi_q(i(g_C), s) = 1$ unless $q = p_i$ for some i. In that case

$$\begin{aligned}\Phi_{p_i}(i(g_C), s) &= \Phi_{p_i}(i(p_i^{e_i}), s) = \chi_{p_i}(p_i^{e_i})\, p^{-e_i(s+1/2)}\\ &= \chi(\mathfrak{P}_i^{e_i})(N\mathfrak{P}_i^{e_i})^{-1/2}\, (N\mathfrak{P}_i)^{-e_i s}.\end{aligned}$$

So

$$\tilde{\eta}(\mathfrak{A})\, \Phi^*(i(g_C), s) = \tilde{\eta}(\mathfrak{A})\, \Lambda(2s+1, \varepsilon) \prod \Phi_q(i(g_C), s) = \frac{\mu(\mathfrak{A})}{a^{k+1}}\, a^{-s} \Lambda(2s+1, \varepsilon).$$

On the other hand, one has by Propositions 2.2–2.5 and 3.1,

$$\begin{aligned}\tilde{\eta}(\mathfrak{A}) \prod_{q \leqslant \infty} W_{0,q}^*(i(g_C), s) &= \Lambda(2s, \varepsilon)(-i 2^{2s} p^s)\, \frac{-i}{\sqrt{p}}\, p^{-s}\, \frac{\mu(\mathfrak{A})}{a^{k+1}}\, a^s \prod_{j=1}^{k} \frac{j-s}{j+s}\\ &= -2^{2s} a^s p^{-1/2}\, \frac{\mu(\mathfrak{A})}{a^{k+1}}\, \Lambda(2s, \varepsilon) \prod_{j=1}^{k} \frac{j-s}{j+s}.\end{aligned}$$

So

$$\begin{aligned}\Lambda_0(s, \mu, C) &= \frac{1}{2}\, \tilde{\eta}(\mathfrak{A}) \left[\Phi^*(i(g_C), s) + \prod_{q \leqslant \infty} W_{0,q}^*(i(g_C), s) \right]\\ &= \frac{\mu(\mathfrak{A})}{2a^{k+1}} \left[a^{-s} \Lambda(2s+1, \varepsilon) - 2^{2s} a^s p^{-1/2} \Lambda(2s, \varepsilon) \prod_{j=1}^{k} \frac{j-s}{j+s} \right].\end{aligned}$$

The functional equation

$$p^{(1+s)/2} \Lambda(s, \varepsilon) = p^{(2-s)/2} \Lambda(1-s, \varepsilon)$$

implies

$$p^{-1/2} \Lambda(2s, \varepsilon) = p^{-2s} \Lambda(1-2s, \varepsilon).$$

So

$$\Lambda_0(s, \mu, C) = \frac{2^s \mu(\mathfrak{A})}{2a^{k+1} p^s \prod_{j=1}^{k} (j+s)} \left(G\left(\frac{p}{2a}, s\right) - G\left(\frac{p}{2a}, -s\right) \right)$$

as desired. It is obvious from this formula that

$$\Lambda_0'(0,\mu,C)=\frac{\mu(\mathfrak{A})\,G(p/2a,0)}{a^{k+1}k!}\,\frac{G'(p/2a,0)}{G(p/2a,0)}$$
$$=\frac{\mu(\mathfrak{A})\,\Lambda(1,\varepsilon)}{a^{k+1}}\left[\log\frac{p}{2a}+2\frac{\Lambda'(1,\varepsilon)}{\Lambda(1,\varepsilon)}+\sum_{j=1}^{k}\frac{1}{j}\right].$$

Now the final formula follows from the well known number theory fact

$$\Lambda(1,\varepsilon)=\frac{h_p}{\sqrt{p}}. \tag{3.10}$$

Proof of Theorem 0.1. By Propositions 3.4 and 3.5, we have

$$\Lambda'(0,\mu,C)=\frac{\mu(\mathfrak{A})}{\sqrt{p}\,a^{k+1}}\left[h_p\left(\log\frac{p}{2a}+2\frac{\Lambda'(1,\varepsilon)}{\Lambda(1,\varepsilon)}+\sum_{j=1}^{k}\frac{1}{j}\right)\right.$$
$$-\frac{2}{\sqrt{p}}\sum_{n>0}a_n C_k\left(-\frac{2\pi n}{a\sqrt{p}}\right)e_p(n\tau_{\mathfrak{A}})$$
$$\left.-\frac{2}{\sqrt{p}}\sum_{n<0}\rho(-n)\,\beta_k\left(-\frac{2\pi n}{a\sqrt{p}}\right)e_p(n\tau_{\mathfrak{A}})\right].$$

It is clear from this formula that

$$\Lambda'(0,\mu\xi,C)=\xi(C)\,\Lambda'(0,\mu,C)$$

for every ideal class character ξ. Combining this with (0.2) and (3.6), one sees that

$$L'(k+1,\mu,C)=\pi\Lambda'(0,\mu,C). \tag{3.11}$$

This proves the main formula.

Remark 3.6. It is easy to see by Proposition 3.1 that

$$\Lambda(s,\mu\xi,C)=\xi(C)\,\Lambda(s,\mu,C)$$

for every ideal class character ξ of E. Combining this with (1.10), we see that $\Lambda(s,\mu,C)/L(2s+1,\varepsilon_\infty)$ satisfies (0.2) with a proper shift on s. So

$$\Lambda(s,\mu,C)=L(2s+1,\varepsilon_\infty)\,L(s+k+1,\mu,C). \tag{3.12}$$

4. A VARIANT

Define two functions

$$\theta_k(\tau) = h_p + 2 \sum_{n>0} \rho(n)\, C_k(-4\pi n\, \mathrm{Im}(\tau))\, e(n\tau) \tag{4.1}$$

and

$$\phi_k(\tau) = a_0(\tau) - 2 \sum_{n>0} a_n\, C_k(-4\pi n\, \mathrm{Im}(\tau))\, e(n\tau)$$
$$-2 \sum_{n<0} \rho(-n)\, \beta_k(-4\pi n\, \mathrm{Im}(\tau))\, e(n\tau). \tag{4.2}$$

Here

$$a_0(\tau) = h_p \left(\frac{3}{2} \log p + \log \mathrm{Im}(\tau) + 2 \frac{\Lambda'(1, \varepsilon)}{\Lambda(1, \varepsilon)} + \sum_{j=1}^{k} \frac{1}{j} \right). \tag{4.3}$$

Both functions are closely related to the Taylor expansion of the well-known Eisenstein series

$$E_k(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \varepsilon(d)(c\tau + d)^{-2k-1}\, |c\tau + d|^{-2s}$$

at the symmetric center [Ya4]. When $k=0$, ϕ_0 is the modular form constructed in [KRY] (up to a sign).

First we note that Theorem 0.1 can be rewritten as

$$L'(k+1, \mu, C) = \frac{\pi \mu(\mathfrak{A})}{\sqrt{p}\, N(\mathfrak{A})^{k+1}}\, \phi_k \left(\frac{\tau_{\mathfrak{A}}}{p} \right) \tag{4.4}$$

for every ideal class C of $E = \mathbb{Q}(\sqrt{-p})$. In particular,

$$L'(k+1, \mu, \text{trivial}) = \frac{\pi}{\sqrt{p}}\, \phi_k \left(\frac{1}{2} + \frac{1}{2\sqrt{p}}\, i \right). \tag{4.5}$$

This is the first formula in Theorem 0.2 when $k=0$. The purpose of this section is to prove

THEOREM 4.1. *Let the notation be as above. Then*

$$L'(k+1, \mu, \text{trivial}) = \frac{4\pi}{\sqrt{p}} \left(\phi_k \left(\frac{2i}{\sqrt{p}} \right) - \theta_k \left(\frac{i}{\sqrt{p}} \right) \log 2 \right).$$

That is,

$$\frac{\sqrt{p}}{4\pi} L'(k+1, \mu, \text{trivial}) = h_p \left(\log p + 2 \frac{\Lambda'(1, \varepsilon)}{\Lambda(1, \varepsilon)} + \sum_{j=1}^{k} \frac{1}{j} \right)$$
$$-2 \sum_{n>0} a_n C_k \left(-\frac{8\pi n}{\sqrt{p}} \right) e^{-4\pi n/\sqrt{p}}$$
$$-2 \sum_{n<0} \rho(-n)\, \beta_k \left(-\frac{8\pi n}{\sqrt{p}} \right) e^{-4\pi n/\sqrt{p}}$$
$$-2 \log 2 \sum_{n>0} \rho(n)\, C_k \left(-\frac{4\pi n}{\sqrt{p}} \right) e^{-2\pi n/\sqrt{p}}.$$

When $k=0$, this gives the second formula in Theorem 0.2.

LEMMA 4.2. *One has*

$$\phi_k(\tfrac{1}{2}+\tau) = 2\phi_k(4\tau) - \phi_k(\tau) - 4\theta_k(2\tau) \log 2.$$

Proof. Define $a_n = \rho(n) = 0$ when n is not an integer. Since 2 is inert in $E = \mathbb{Q}(\sqrt{-p})$, one has for $n > 0$

$$(-1)^n a_n = -a_n + 2a_{n/4} + 4 \log 2\rho \left(\frac{n}{2} \right). \tag{4.6}$$

Indeed, when n is odd, it is trivial. When $n \equiv 2 \bmod 4$,

$$(-1)^n a_n = a_n = 2 \log 2\rho \left(\frac{n}{2} \right) = a_{n/4} + 2 \log 2\rho \left(\frac{n}{2} \right).$$

When $n \equiv 0 \bmod 4$, one has

$$(-1)^n a_n = a_n = a_{n/4} + 2 \log 2\rho \left(\frac{n}{8} \right) = a_{n/4} + 2 \log 2\rho \left(\frac{n}{2} \right).$$

For $n < 0$, one has similarly

$$(-1)^n \rho(-n) = -\rho(-n) + 2\rho \left(-\frac{n}{4} \right).$$

Finally,

$$a_0(4\tau) = 2 \log 2 + a_0(\tau) = 2 \log 2 + a_0(\tfrac{1}{2}+\tau).$$

Now the lemma follows from a simple calculation.

For $\tau = x + (y/2\sqrt{p})\, i \in \mathfrak{H}$, the upper half plane, set

$$g_\tau = n(x)\, m(\sqrt{y}) \in G(\mathbb{R}). \tag{4.7}$$

We remark that $g_\tau(-1/2\delta) = \tau$.

LEMMA 4.3. *Let the notation be as above. Then*

$$\phi_k\left(\frac{1}{2} + \tau\right) = \frac{\sqrt{p}}{2\sqrt{y}}\, E^{*,\prime}(g_\tau, 0).$$

Proof (Sketch). A simple calculation gives

$$W^*_{d,\infty}(g_\tau, s) = y^{1/2-s} e(dx)\, W^*_{dy,\infty}(s).$$

So one has for $d \neq 0$,

$$E^*_d(g_\tau, s) = E^*_d(1, s)\, \frac{W^*_{dy,\infty}(s)}{W^*_{d,\infty}(s)}\, y^{1/2-s} e(dx)$$

and

$$E^{*,\prime}_d(g_\tau, 0) = E^{*,\prime}_d(1, 0) \lim_{s \mapsto 0} \frac{W^*_{dy,\infty}(s)}{W^*_{d,\infty}(s)}\, y^{1/2} e(dx).$$

Recall that (formula (3.4))

$$E^{*,\prime}_d(1, 0) = 2\Lambda'(0, \mu, \text{trivial}).$$

Now Propositions 2.3 and 3.4 give (the ideal class C is trivial here)

$$E^{*,\prime}_d(g_\tau, 0) = \begin{cases} \dfrac{4\sqrt{y}}{\sqrt{p}}\, (-1)^{n-1}\, a_n C_k(-4\pi n\, \mathrm{Im}(\tau))\, e(n\tau) & \text{if} \quad n > 0, \\[2ex] \dfrac{4\sqrt{y}}{\sqrt{p}}\, (-1)^{n-1}\, \rho(-n)\, \beta_k(-4\pi n\, \mathrm{Im}(\tau))\, e(n\tau) & \text{if} \quad n < 0. \end{cases}$$

Calculation similar to the proof of Proposition 3.5 gives

$$E^*_0(g_\tau, s) = \frac{2^s y^{1/2}}{p^s \prod_{j=1}^k (j+s)} \left(G\left(\frac{py}{2}, s\right) - G\left(\frac{py}{2}, -s\right)\right).$$

So

$$E_0^{*,\prime}(g_\tau, 0) = 2y^{1/2}\Lambda(1,\varepsilon)\left(\log\frac{py}{2} + 2\frac{\Lambda'(1,\varepsilon)}{\Lambda(1,\varepsilon)}\right) + \sum_{j=1}^{k}\frac{1}{j}$$
$$= \frac{2\sqrt{y}}{\sqrt{p}}\, a_0(\tau).$$

Therefore,

$$E^{*,\prime}(g_\tau, 0) = \frac{2\sqrt{y}}{\sqrt{p}}\,\phi_k\left(\frac{1}{2}+\tau\right),$$

as desired.

PROPOSITION 4.4. *One has the functional equation*

$$\phi_k\left(-\frac{1}{p\tau}\right) = -\sqrt{p}\,|\tau|\left(\frac{-\delta\tau}{|\delta\tau|}\right)^{2k+1}\phi_k(\tau).$$

Proof (Sketch). By Lemma 4.3, this proposition follows from the following trivial functional equation plus a long tedious calculation on both sides.

$$E^*(\gamma_f^{-1}g, s) = E^*(\gamma_\infty g, s) \tag{4.7}$$

for any $\gamma \in G(\mathbb{Q})$. Here γ_f and γ_∞ are the finite and infinite parts of γ, viewed as an element in $G(\mathbb{A})$. Indeed, taking $g = g_\tau$ and $\gamma = n(\frac{1}{2})\,\alpha n(-\frac{1}{2})$ with

$$\alpha = \begin{pmatrix} 0 & -\frac{1}{\delta} \\ -\delta & 0 \end{pmatrix}$$

and computing both sides of (4.7), one gets

$$-E^{*,\prime}(g_\tau, 0) = \left(\frac{|-\delta(\tau-1/2)|}{-\delta(\tau-1/2)}\right)^{2k+1} E^{*,\prime}(g_{\tau'}, 0), \tag{4.8}$$

with $\tau' = \frac{1}{2} - 1/p(\tau - \frac{1}{2})$. Replacing τ by $\frac{1}{2}+\tau$, this is exactly what we claimed in this proposition by Lemma 4.3. Here is a sketch to derive (4.8) from (4.7). First,

$$\gamma_\infty\, g_\tau = t g_{\tau'} i(-t^2)$$

with $t=-\delta(\tau-\frac{1}{2})/|\delta(\tau-\frac{1}{2})|$, and so

$$W^*_{d,\infty}(\gamma_\infty g_\tau, s)=\left(\frac{|-\delta(\tau-1/2)|}{-\delta(\tau-1/2)}\right)^{2k+1} W^*_{d,\infty}(g_{\tau'}, s)$$

and

$$E^*(\gamma_\infty g_\tau, s)=\left(\frac{|-\delta(\tau-1/2)|}{-\delta(\tau-1/2)}\right)^{2k+1} E^*(g_{\tau'}, s).$$

On the other hand, when q is nonsplit,

$$\gamma_q^{-1}=n(x)\, m(y)\, i(g)$$

with

$$y=\frac{2}{1-\delta}, \qquad g=-\frac{1-\delta}{1+\delta}, \qquad x=\frac{5+p}{2(1+p)}.$$

This implies that

$$W^*_{d,q}(\gamma_q^{-1}, s)=(\chi\tilde{\eta})_q\left(\frac{2}{1-\delta}\right)\tilde{\eta}_q(\delta)\ W^*_{d,q}(s).$$

When q is split, $\delta=(x_q, -x_q)$ and

$$\gamma_q^{-1}=n(x)\,\mathrm{diag}(y_1, y_2)\, i(g)$$

with

$$y_1=\frac{2}{1-x_q}, \qquad y_2=\frac{1+x_q}{2}, \qquad g=-\frac{1-x_q}{1+x_q}.$$

So (after a long calculation using results in Section 2)

$$W^*_{d,q}(\gamma_q^{-1}, s)=W^*_{d,q}(s).$$

So one has for $d\neq 0$

$$E^*_d(\gamma_f^{-1} g_\tau, s)=\prod_{q \text{ nonsplit}}(\chi\tilde{\eta})_q\left(\frac{2}{1-\delta}\right)\tilde{\eta}_q(\delta)\, E^*_d(g_\tau, s)=-E^*_d(g_\tau, s).$$

The same is true for $d=0$. Therefore (4.7) implies

$$-E^*(g_\tau, s)=\left(\frac{|-\delta(\tau-1/2)|}{-\delta(\tau-1/2)}\right)^{2k+1} E^*(g_{\tau'}, s).$$

Taking the derivative on both sides at $s=0$, one gets (4.8).

Proof of Theorem 4.1. Now the proof of Theorem 4.1 becomes easy. Indeed, taking $\tau=(1/2\sqrt{p})\,i$ in Proposition 4.4, one gets

$$\phi_k\left(\frac{1}{2\sqrt{p}}\,i\right)=-2\phi_k\left(\frac{2}{\sqrt{p}}\,i\right).$$

Now (4.5) and Lemma 4.2 imply

$$\begin{aligned}\frac{\sqrt{p}}{\pi}L'(k+1, \mu, \text{trivial}) &= \phi_k\left(\frac{1}{2}+\frac{i}{2\sqrt{p}}\right)\\ &= 2\phi_k\left(\frac{2i}{\sqrt{p}}\right)-\phi_k\left(\frac{i}{2\sqrt{p}}\right)-4\theta_k\left(\frac{i}{\sqrt{p}}\right)\log 2\\ &= 4\phi_k\left(\frac{2i}{\sqrt{p}}\right)-4\theta_k\left(\frac{i}{\sqrt{p}}\right)\log 2.\end{aligned}$$

So

$$L'(k+1, \mu, \text{trivial})=\frac{4\pi}{\sqrt{p}}\left(\phi_k\left(\frac{2i}{\sqrt{p}}\right)-\theta_k\left(\frac{i}{\sqrt{p}}\right)\log 2\right)$$

as claimed.

Finally, we want to mention that Miller and the author [MY] have just proved that the central derivative $L'(1, \mu)\neq 0$ for every canonical Hecke character μ of $\mathbb{Q}(\sqrt{-p})$ of weight one. This implies the inequality

$$\phi_0\left(\frac{2i}{\sqrt{p}}\right)>\theta_0\left(\frac{i}{\sqrt{p}}\right)\log 2, \tag{4.9}$$

for $p\equiv 3 \bmod 8$. It would be interesting to prove this inequality directly, which would give an independent proof of $L'(1, \mu)\neq 0$.

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